

Coloring Graphs to Produce Properly Colored Walks

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Abstract

For a connected graph, we define the proper-walk connection number as the minimum number of colors needed to color the edges of a graph so that there is a walk between every pair of vertices without two consecutive edges having the same color. We show that the proper-walk connection number is at most three for all cyclic graphs, and at most two for bridgeless graphs. We also characterize the bipartite graphs that have proper-walk connection number equal to two, and show that this characterization also holds for the analogous problem where one is restricted to properly colored paths.

1 Introduction

We consider the problem of coloring the edges of a graph so that it is possible to get between every pair of vertices without two consecutive edges having the same color. Obviously, this can be achieved by giving every edge a different color, and indeed by any proper coloring of the edges. So the real question is what is the minimum number of colors one needs.

Borožan et al. [2] introduced this problem for paths. In particular, for a connected graph G , they defined the *proper connection number* as the minimum number of colors that one needs so that there is a properly colored path between every two vertices. For example, they showed that the parameter is at most 3 for any block. Also, if a graph has a Hamiltonian path, then the parameter is at most 2 [1], and thus almost surely this holds for a random graph [3]. For a recent survey, see [5].

We consider here the analogous concept for walks. For a connected graph G , we define the *proper-walk connection number* $pW(G)$ as the minimum number

of colors if one is allowed any properly colored walk. For symmetry, we will use $pP(G)$ to denote the proper connection number. Trivially, $pW(G) \leq pP(G)$.

We proceed as follows. In Section 2 we show that for any connected cyclic graph the proper-walk connection number is at most three, and in Section 3 we characterize the bipartite graphs that have proper-walk and proper connection numbers two. Thereafter, we show in Section 4 that the parameter is two for any graph with two disjoint odd cycles and in Section 5 that the parameter is two for any bridgeless graph. In Section 6 we provide some thoughts on the general case. We conclude with a comment about the directed version and some thoughts for future work.

2 An Upper Bound

It is immediate that a properly colored walk cannot use the same edge twice in succession. It follows that, in a tree, every properly colored walk is a path. As observed in [1], for the property in trees, one needs the edges of the tree to be properly colored, and thus:

Observation 1 *If T is a tree with maximum degree Δ , then $pW(T) = pP(T) = \Delta$.*

We present next a general upper bound on the proper-walk connection number of cyclic graphs.

Theorem 1 *Let G be a connected graph that is not a tree. Then $pW(G) \leq 3$.*

PROOF. We may assume that G is unicyclic (else take suitable spanning subgraph). Consider the cycle C . Take any proper coloring of the cycle C . For every vertex v of the cycle, it is incident with two colors in the cycle; so let all other edges incident with v have the third color. Color the remaining edges so that for every vertex w not on the cycle, the path J_w from w to the closest vertex of C is properly colored.

There is a properly colored walk between every pair u and v of vertices. For example, if both u and v are off the cycle, then use J_u to get to the cycle, go

around the cycle to the vertex closest to v , and then use J_v in reverse to get to v .
QED

Figure 1 gives an example of a graph G where $pW(G) = 3$. (For a proof of this, see Theorem 7.)

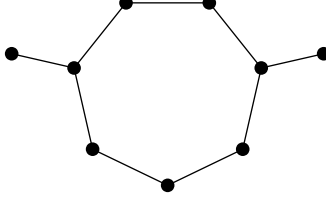


Figure 1. A graph G such that $pW(G) = 3$

Note that the complete graph has $pW(G) = pP(G) = 1$, while noncomplete graphs have $pW(G) \geq 2$. So the big question is: for which graphs is $pW(G) = 2$?

3 Bipartite Graphs

We next determine which bipartite graphs G have $pW(G) = 2$.

For a graph G , define $\mathcal{M}(G)$ as the set of all maximal 2-edge-connected subgraphs. Equivalently, these are the individual components that result if one removes all bridges. (Note that these can be isolated vertices.)

Theorem 2 *Let G be a connected bipartite graph. Then $pW(G) = 2$ if and only if every subgraph in $\mathcal{M}(G)$ is incident with at most two bridges.*

PROOF. (1) Assume that every subgraph in $\mathcal{M}(G)$ is incident with at most two bridges. Let H be a nontrivial subgraph in $\mathcal{M}(G)$. Then H is 2-edge connected. By Robbins' Theorem [7], such a graph has a strongly connected orientation. Give the vertices of the subgraph H its bipartite coloring; then color each edge/arc of H by the color of its head. It follows that all directed walks in the orientation of H alternate colors. And within the undirected H , each pair of vertices is joined by a properly colored walk that starts with any designated color or ends with any designated color (by either following the arcs or going against the arcs). Do the same for all nontrivial subgraphs in $\mathcal{M}(G)$.

Now, consider the graph F obtained from G by contracting each (nontrivial) subgraph H in $\mathcal{M}(G)$ to a single vertex c_H . Note that each edge in F comes from a bridge in G ; in particular, F is a forest of maximum degree at most 2, so that every component of F is a path. For each bridge e of G , let e' be the corresponding edge in F .

We color each component of F as follows. Start at a leaf-edge and give it either color. For subsequent edges, suppose that edge e' is colored and we need to color adjacent edge f' . If edges e' and f' have common end c_H , then let v_e be the end of e in H and similarly with v_f . If v_e and v_f are in the same partite set of G , then give edges e and f different colors; and if v_e and v_f are in different partite sets of G , then give edges e' and f' the same color. Finally, color each bridge e of G by the color of its corresponding edge e' in F .

We claim this coloring of the H 's and bridges has the desired property. For, within any bridgeless subgraph H , the property holds. And if one enters the subgraph H on a bridge of one color, one can continue with the other color. If one needs to leave H again, one will reach the exit vertex with the appropriate edge color.

(2) Assume that G has a suitable 2-coloring. Since G is bipartite, every closed walk has the same parity. So assume a properly colored walk enters a subgraph H of $\mathcal{M}(G)$ along bridge b_1 to vertex v_1 and exits H along bridge b_2 from vertex v_2 (with $v_1 = v_2$ allowed). Then b_1 and b_2 must have color determined by the parity of the distance between v_1 and v_2 . That is, bridges b_1 and b_2 have the same color if and only if v_1 and v_2 are in different partite sets in G .

So suppose there are three bridges b_1, b_2, b_3 incident with (not necessarily distinct) vertices v_1, v_2, v_3 of H . Without loss of generality, v_1 and v_2 are in the same partite set X . Thus b_1 and b_2 need different colors. But then if v_3 is in X , the bridge b_3 needs a color different from both b_1 and b_2 ; and if v_3 is in the other partite set, then b_3 needs to be the same as both b_1 and b_2 ; in each case an impossibility. QED

It turns out that the above characterization also holds for the proper connection number. For, in a bipartite graph, all closed walks have even length. Thus, if the edges are 2-colored, then there is a properly colored walk between

two vertices if and only if there is a properly colored path between them. That is:

Theorem 3 *Let G be a connected bipartite graph. Then $pP(G) = 2$ if and only if every subgraph in $\mathcal{M}(G)$ is incident with at most two bridges.*

It was known that $pP(G) = 2$ for bridgeless bipartite graphs [2].

4 Disjoint Odd Cycles

We now consider the general problem of which graphs G have $pW(G) = 2$.

Theorem 4 *If a connected graph G has two edge-disjoint odd cycles, then $pW(G) = 2$.*

PROOF. Let C_1 and C_2 be edge-disjoint odd cycles. If they are also vertex-disjoint, let P be a shortest path joining them; say P starts with vertex u_1 in C_1 and ends at u_2 in C_2 . If the cycles have a vertex in common, then let $u_1 = u_2$ be such a vertex. Let H be the subgraph consisting of C_1 , C_2 , and P if needed.

Now, color the two edges of C_1 incident with u_1 red; then color the remaining edges of C_1 alternating red and blue so that u_1 is the only vertex not incident with an edge of each color. Further, if P exists, color the edges of P alternating colors so that the edge incident with u_1 is blue. Now, if P has even length or the cycles had a vertex in common, color the two edges of C_2 incident with u_2 blue; then color the remaining edges of C_2 alternating red and blue so that u_2 is the only vertex not incident with an edge of each color. On the other hand, if P has odd length, then proceed similarly, except that the two edges of C_2 incident with u_2 are colored red. Note that this implies that between every pair of (not necessarily distinct) vertices in H there is a properly colored walk that starts and finishes with any prescribed colors.

Now consider the vertices not in H . By choosing a spanning subgraph if needed, one may assume that for each vertex v not in H there is a unique path J_v from v to H . Color the remaining edges such that each J_v is properly colored. See Figure 2.

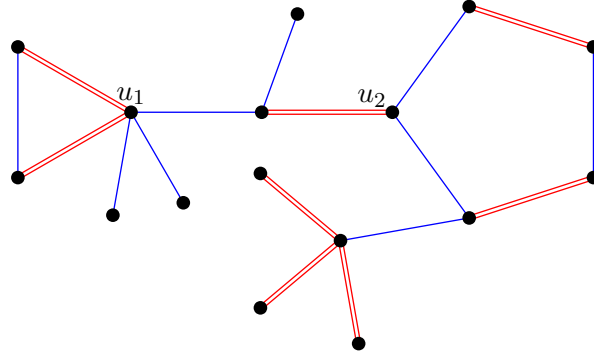


Figure 2. *A coloring of a graph with two disjoint odd cycles*

We claim that the coloring has the desired property. To get between any two vertices v and w in G , use the alternating path J_v to get to H if necessary, go around H in the appropriate direction, and then use the alternating path J_w in reverse if needed. QED

Our focus is on simple graphs, but we consider in passing what happens if the graph has loops. It is immediate from the above that if the graph has two loops then $pW(G) = 2$, as one can treat the loops as odd cycles. But actually, $pW(G) = 2$ for any graph with a loop. For, one can color the loop blue say, the edges incident with the loop red, and then alternate colors away from the loop. There is a properly colored walk between every pair of vertices by going via the loop.

5 Bridgeless Graphs

In this section, we show that $pW(G) = 2$ for all connected graphs G without bridges.

We will need the following simple observation.

Observation 2 *Let P be an induced path. If there is an odd cycle that shares at least one edge with the path P , then there exists a nontrivial path S that is internally disjoint from P and creates an odd cycle with P .*

PROOF. Let C be any odd cycle that shares an edge with P . Consider the vertices of $C \cap P$. Since P is induced, there must be at least one vertex in C not

on P . Since C and P share an edge, there are at least two vertices in $C \cap P$. Now, partition the edges of C not in P into segments, where the ends of a segment are in P and internal vertices of each segment are not in P . If every segment creates an even cycle with P , then the result is bipartite, a contradiction. So some segment creates an odd cycle with P , as required. QED

We define a ***theta-graph*** as a graph that is formed by taking a cycle C of even length (called the outer cycle) and a path P (called the inverter) and identifying the ends of the path P with two vertices u and v of the cycle C such that the result is nonbipartite. See Figure 3 for an example.

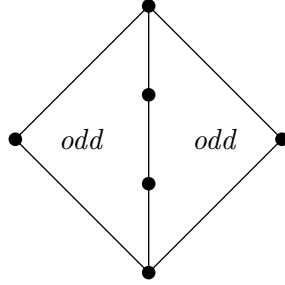


Figure 3. A *theta-graph*

Theorem 5 *There does not exist a 2-connected graph G such that $pW(G) = 3$.*

PROOF. Suppose block G has $pW(G) = 3$. We saw above (Theorem 2) that G cannot be bipartite. Also, we saw (Theorem 4) that G does not contain two edge-disjoint odd cycles.

Consider some odd cycle of the graph G . If it is a hamilton cycle, then it is easily seen that $pW(G) = 2$. So assume there is some vertex not on this cycle. By 2-connectedness, we can find two disjoint paths from this vertex to the cycle, ending at vertices u and v say. That is, we have three internally disjoint $u-v$ paths such that the result is not bipartite. Two of these paths have the same parity; choose them to be the outer cycle, and the other path to be the inverter. That is, the result is a theta-graph.

Out of all theta-subgraphs,

choose the theta-subgraph where the inverter P is as short as possible.

Let C be the outer cycle of the chosen theta-graph.

Claim. *The graph $G - C$ is bipartite.*

PROOF. Suppose there is an odd cycle in $G - C$. Since there are not two edge-disjoint odd cycles in G , that odd cycle must share an edge with (the interior of) P . Then by Observation 2, there is segment S in $G - (C \cup P)$ that joins two vertices of the interior of P but is otherwise disjoint from P and creates an odd cycle with P . This segment S combined with P and either half of C provides a theta-graph with a shorter inverter, which contradicts our choice of theta-subgraph. QED

Now, color the graph G as follows. Color the theta-graph such that the outer cycle C is properly colored, as is the inverter P . Without loss of generality, assume that C is drawn so that every properly colored walk leaving the inverter proceeds clockwise on the outer cycle.

Partition the remaining vertices into two sets: let A be those vertices that can reach the outer cycle C without going through an internal vertex of P , and let B be those that cannot. For each vertex w of A , retain one path J_w to C . Color the edge of J_w incident with C such that one can go across that edge and proceed counter-clockwise around the outer cycle. Color the remaining edges of the path J_w so that it is properly colored.

Finally, consider B . By 2-connectivity, there are two internally disjoint paths from any vertex $w \in B$ to the theta-graph. By the definition of B , both these paths must meet the theta-graph at an internal vertex of P . Let P' be P minus u and v . Orient the path P' from u to v . We will create an oriented spanning subgraph H of the graph induced by $P' \cup B$ such that: for every vertex w of B , there is an oriented walk Q_w from w to P' and an oriented walk R_w from P' to w , so that the end of Q_w occurs before the start of R_w on P' .

For the first vertex w of B , we have the two internally disjoint paths that go to P' , and can orient these so that the result is an oriented cycle. For further vertices w of B , take the two disjoint paths from w and cut each when it reaches a vertex that is already in H . Say we have internally-disjoint paths L_1 and L_2 from w to h_1 and h_2 . If necessary, re-order h_1 and h_2 so that the end of Q_{h_1}

occurs before the start of R_{h_2} on P' . Then orient L_1 away from w and orient L_2 towards w ; the desired Q_w is L_1 followed by Q_{h_1} and the desired R_w is R_{h_2} followed by the reverse of L_2 . Note too that this orientation works for all internal vertices on L_1 and L_2 .

Now, consider the strong components H_i of H . These entail disjoint segments of P' . We know H is bipartite, so each H_i is too. As in Theorem 2, we can color the edges of H_i so that one can get between every two vertices of H_i with a properly colored walk that respects the orientation. In particular, this edge-coloring properly colors the segment of P' , and thus extends the coloring of the theta-graph. See Figure 4 for an example.

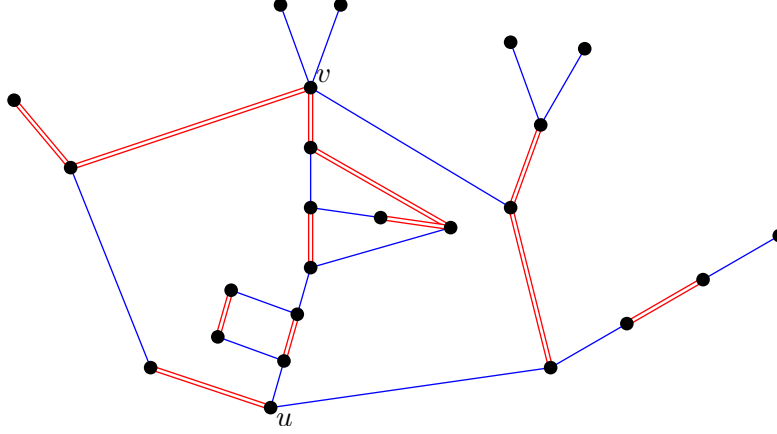


Figure 4. *Coloring of theta-graph and rest of graph*

The resultant coloring has a properly colored walk between any pair of vertices. For example, to get from a vertex w_1 of A to another vertex w_2 of A , follow J_{w_1} , go counter-clockwise around the outer cycle, over the inverter, clockwise around the outer cycle, and then use J_{w_2} in reverse. To get from a vertex w_1 of A to a vertex w_3 of B , follow J_{w_1} , go counter-clockwise around the outer cycle to u , over the inverter, and then along R_{w_3} . QED

From the above result, the question of bridgeless graphs is easily resolved:

Theorem 6 *If G is a connected bridgeless graph, then $pW(G) \leq 2$.*

PROOF. Assume G is bridgeless but not 2-connected. Consider the blocks of G . If any two of these are nonbipartite, then there are two edge-disjoint odd cycles,

and the result follows from Theorem 4. If all the blocks are bipartite, then the result follows from Theorem 2. So assume that exactly one block, say H , is not bipartite.

By the above theorem, that block H can be colored with two colors to have a properly colored walk between every pair of vertices in H . Color all remaining blocks properly, as in Theorem 2. We claim the resultant coloring has the desired property. To find a properly colored walk between vertices u and v , let u' be the vertex of H nearest to u and v' the vertex of H nearest to v . Then find the properly colored walk between u' and v' . This can be extended to a properly colored walk between u and v , since there is a walk from u to u' ending with any prescribed color, and a walk from v' to v starting with any desired color. QED

6 Unicyclic Graphs

It is unclear what happens in general in graphs with bridges. For example, consider the collection \mathcal{G} of graphs formed by taking an odd cycle and adding feet to some of the vertices of the cycle. (By adding a foot we mean adding a new vertex and joining it to exactly one vertex of the cycle.)

Theorem 7 *Let G be a graph of \mathcal{G} . Then $pW(G) = 2$ if and only if there are three consecutive vertices u, v, w on the cycle such that u is adjacent to at most one foot, w is adjacent to at most one foot, and all vertices other than u, v, w are incident with no feet.*

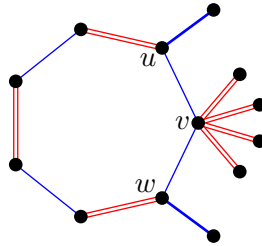


Figure 5. A graph in \mathcal{G} with $pW = 2$

PROOF. (1) We first prove that the conditions are necessary for the graph to have $pW = 2$. That is, assume the graph has a 2-coloring such that every vertex can reach every other vertex by a properly colored walk.

Assume for the time being that the odd cycle has length at least 5. Then, up to isomorphism, the odd cycle has a unique 2-coloring such that every vertex can reach every other vertex by properly colored walks. For, if more than one vertex is incident with two edges of the same color, then by parity, there are at least three such vertices, and each is a break on the cycle, and one pair of breaks must leave a nontrivial piece, a contradiction.

So let v be the unique vertex on the cycle incident with two edges of the same color, with neighbors u and w on the cycle. Suppose there is a foot attached to a vertex x that is neither u , v , nor w . Then the two edges of the cycle incident with x have different colors, and so any walk from the foot can proceed in only one direction around the cycle, and gets stopped at v without reaching all the vertices. Thus, all feet must be attached to one of u , v , or w .

Consider a foot incident with u . In order for it to reach all vertices, the edge incident with it must have the same color as the uv edge. It follows that the foot is unique, since otherwise the two feet would not be able to reach each other.

Finally, return to the odd cycle being the triangle. If the triangle has exactly one vertex v incident with two edges of the same color, then by the same argument as above, the other two vertices of the triangle can be incident with at most one foot each. Further, if the triangle is monochromatic, then it is easy to see that each vertex of the cycle is incident with at most one foot.

(2) We second prove that the conditions are sufficient. Color the cycle such that v is incident with two edges of the same color and every other vertex sees both colors. Color the leaf incident with u and/or w with the same color as the uv edge; color all leaves incident with v with the other color. It is easily checked that this coloring has the desired property. QED

7 Directed Graphs

For a strongly connected digraph, one can define the proper-walk connection number as in the undirected case. This idea was recently introduced for paths by Magnant et al. [6]. They showed that:

Theorem 8 [6] *Let D be a strongly-connected digraph. Then $pP(D) \leq 3$.*

This is sharp, even for the proper-walk case, since an odd cycle needs three colors; that is, $pW(D) = pP(D) = 3$ if D is an odd cycle.

We note that the two parameters can be different. That is, there are digraphs with $pW(D) = 2$ and $pP(D) = 3$. For example, take two disjoint directed triangles and identify one vertex of each. See Figure 6.

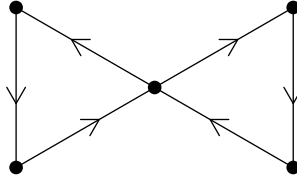


Figure 6. A graph where $pW(D) = 2$ and $pP(D) = 3$

Another direction it to add loops. If one adds loops at all vertices, then one needs only two colors (color all original arcs one color and all loops a second color).

8 Conclusion

We proved that every connected graph has proper-walk connection number at most three, and showed that it is two for some families. One natural open problem is the complexity of recognizing which graphs have the parameter 2. Is there a polynomial-time algorithm, or is it NP-hard? Note that it is easy to check using a breadth-first-search whether a given coloring has a properly colored walk between two vertices.

Other directions of interest include the question where some of the edges of the graph are already colored. For example, Kézdy and Wang [4] asked when

one could complete a 2-coloring such that there is an alternating path between two specified vertices. One could also insist on stronger properties; for example, that every pair of vertices is in a properly colored cycle, or closed walk.

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